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A LINE-FORCE LOADING ON THE SURFACE OF A NONLOCAL ELASTIC HALF-INFINITE MEDIUM

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Abstract—Transmission of a concentrated force into a half-infinite nonlocal elastic medium is examined. A nonlocal concentrated force boundary condition which is different from that of Nowinski is derived from the nonlocal principle. The nonlocal stress field is obtained analytically. The result eliminates the stress singularity which occurs in the classical stresses and also the nonlocal stresses derived by Nowinski. A theoretical cut-off strength P_c is discussed.

INTRODUCTION

Nowinski (1986) has recently examined the Boussinesq–Flamant problem for a nonlocal elastic half-infinite space, namely, the concentrated line force $P\delta(x_1)$ acting on the boundary plane $x_2 = 0$ of a half-infinite space. His results show that the nonlocal stresses coincide with the classical expressions revealing a singular behaviour at the point of application of the load. Thus, the nonlocal theory, under the assumption adopted by him, does not offer anything new with regard to the stress field. The displacements, however, differ from their classical counterparts, and can only be evaluated approximately. The fourth order approximation found for the Poisson material (namely, $\lambda = \mu$ and Poisson's ratio $\nu = 0.25$) close to the line of loading displays deviations from the classical values amounting to 35%. It is well known that the applications of nonlocal elasticity may eliminate singularities which occur in the classical formulation and some results appear in physically more acceptable forms. For example, the nonlocal stresses at a linear crack tip (Eringen, 1977a; Wang, 1989a) and at the cores of dislocations (Eringen, 1977b), and the self energy between separated dilatation centres (Kovács and Vörös, 1979) do not exhibit classical singularities. Recently, the classical line force problem (a continuous uniform distribution of a point force acting along a line; Dundurs and Hetenyi, 1965) was generalized to the case of nonlocal elasticity (Wang, 1988). The result shows that none of the classical singularities exist in the stresses. Because of the nonlocal effect, the nonlocal line force is a Gaussian function quite different from the Dirac delta function of the classical line force and it satisfies the generalized definition. It is suggested therefore that, in the description of concentrated forces, instead of using the classical Dirac delta function model, nonsingular attenuation function models should be adopted in nonlocal elasticity. In the problem of a line force acting on the surface of a nonlocal elastic half-infinite space, the nonlocal stresses obtained by Nowinski (1986) still exhibit a classical singular behaviour. The nonlocal theory does not make an advance in eliminating classical stress singularies in this problem. We consider the reason to be that the adoption of the classical Dirac delta function model in the concentrated force boundary condition is incorrect for a nonlocal elastic medium. The correct nonlocal concentrated force boundary condition should be given according to the nonlocal principle (Wang, 1988). The present work reconsiders the concentrated line loading problem of nonlocal elasticity under the nonlocal concentrated force boundary conditions given by the above model. The results show that the classical stress singularity at the point of the application of the load is not present in the nonlocal stresses. The displacements are identical to their classical counterparts.

GENERAL EQUATIONS

The treatments in this section are similar to those of Nowinski (1986). The region $x_2 \geqslant 0$ of a Cartesian rectangular coordinate system, $x_1x_2x_3$, is occupied by a linear, homogeneous, isotropic, nonlocal elastic half-infinite medium. A line force of constant intensity, P, distributed along the x_3-axis, acts on the boundary plane $x_2 = 0$ in the direction of the x_2-axis. The equations of equilibrium under the state of a plane strain and the condition of no body forces are

$$t_{11,1}+t_{12,2} = 0$$
$$t_{12,1}+t_{22,2} = 0. \tag{1}$$

The expression of the nonlocal stresses in the Kroener–Eringen form is

$$t_{ij}(x) = \int_A [2\mu'(|x-x'|)e_{ij}(x')+\lambda'(|x-x'|)e_{kk}(x')\delta_{ij}]\,dx'_1\,dx'_2, \tag{2}$$

where $|x-x'| \equiv |x_1-x'_1|, |x_2-x'_2|$, A is the region $(-\infty < x_1 < \infty, x_2 \geqslant 0)$, μ' and λ' are the nonlocal elastic moduli, and e_{ij} is the linear strain tensor. Equation (2) can be written as

$$t_{11} = \int_A [(2\mu'+\lambda')u'_{1,1}+\lambda' u'_{2,2}]\,dx'_1\,dx'_2$$
$$t_{22} = \int_A [(2\mu'+\lambda')u'_{2,2}+\lambda' u'_{1,1}]\,dx'_1\,dx'_2$$
$$t_{12} = \int_A \mu'(u'_{1,2}+u'_{2,1})\,dx'_1\,dx'_2, \tag{3}$$

where u_1 and u_2 are the displacement components. We apply the Fourier transformation

$$\bar{u}(k,x_2) = \int_{-\infty}^{\infty} u(x_1,x_2)\,e^{ikx_1}\,dx_1$$
$$u(x_1,x_2) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{u}(k,x_2)\,e^{-ikx_1}\,dk \tag{4}$$

to eqns (1) and (3) and arrive at the relations

$$-ik\bar{t}_{11}+\bar{t}_{12,2} = 0$$
$$-ik\bar{t}_{12}+\bar{t}_{22,2} = 0 \tag{5}$$

and

$$\bar{t}_{11} = -ik\int_0^{\infty} \{[2\bar{\mu}'(k,|x'_2-x_2|)+\bar{\lambda}'(k,|x'_2-x_2|)]\bar{u}'_1(k,x'_2)$$
$$+\bar{\lambda}'(k,|x'_2-x_2|)\bar{u}'_{2,2}(k,x'_2)\}\,dx'_2$$
$$\bar{t}_{22} = \int_0^{\infty} \{[2\bar{\mu}'(k,|x'_2-x_2|)+\bar{\lambda}'(k,|x'_2-x_2|)]\bar{u}'_{2,2}(k,x'_2)$$
$$-ik\bar{\lambda}'(k,|x'_2-x_2|)\bar{u}'_2(k,x'_2)\}\,dx'_2$$
$$\bar{t}_{12} = \int_0^{\infty} \bar{\mu}'(k,|x'_2-x_2|)[\bar{u}'_{1,2}(k,x'_2)-ik\bar{u}'_2(k,x'_2)]\,dx'_2. \tag{6}$$

The nonlocal elastic moduli can be determined by matching the phonon dispersion curves based on atomic lattice dynamics and experiments with those resulting from nonlocal elasticity (Eringen, 1977a). Several different forms of the nonlocal elastic moduli have been found and applied to various problems (Eringen, 1983). According to the study of Nowinski (1984) on wave propagation in nonlocal elastic media, the following relations between the Fourier transforms of the nonlocal moduli and Lamé constants can be written:

$$\varepsilon(k) \equiv \frac{\bar{\lambda}(k)}{\lambda} = \frac{\bar{\mu}(k)}{\mu} = \frac{\bar{\lambda}(k)+2\bar{\mu}(k)}{\lambda+2\mu} = \frac{\sin^2(ka/2)}{(ka/2)^2}, \tag{7}$$

where a is the atomic spacing between two neighbouring atoms of a perfect lattice, and k is the Fourier transform coefficient and the wave number. With the nonlocal moduli determined from eqn (7), the dispersion curves of one-dimensional plane elastic waves in nonlocal elasticity coincide with those of the Born–Kármán model of the lattice dynamics in the entire Brillouin zone (Eringen, 1977a). Making use of the result in eqn (7), we represent the nonlocal moduli associated with the problem under discussion in the form

$$\begin{aligned} \bar{\mu}'(k,|x_2'-x_2|) &= \bar{\mu}'(k)\delta(|x_2'-x_2|) \\ \bar{\lambda}'(k,|x_2'-x_2|) &= \bar{\lambda}'(k)\delta(|x_2'-x_2|). \end{aligned} \tag{8}$$

This implies consideration of the nonlocal effect in the direction of the x_1-axis. Nowinski (1986) adopts the description of $\delta_n(|x_2'-x_2|)$ [the nth term of an appropriately selected δ-sequence whose limit for $n \to \infty$ is the Dirac delta function $\delta(|x_2'-x_2|)$] in the direction of the x_2-axis and assumes that, for sufficiently large n, the δ_n possess the shifting property characteristic of the Dirac delta function $\delta(|x_2'-x_2|)$. This is, in essence, the same convention. Substitution of eqn (8) into eqns (6) and (5) now gives

$$\bar{t}_{11}(k,x_2) = [-ik(2\mu+\lambda)\bar{u}_1(k,x_2)+\lambda\bar{u}_{2,2}(k,x_2)]\varepsilon(k) \tag{9a}$$

$$\bar{t}_{22}(k,x_2) = [(2\mu+\lambda)\bar{u}_{2,2}(k,x_2)-ik\lambda\bar{u}_1(k,x_2)]\varepsilon(k) \tag{9b}$$

$$\bar{t}_{12}(k,x_2) = \mu[\bar{u}_{1,2}(k,x_2)-ik\lambda\bar{u}_2(k,x_2)]\varepsilon(k) \tag{9c}$$

$$\begin{aligned} \mu\bar{u}_{1,22}-iks(\mu+\lambda)\bar{u}_{2,2}-k^2(2\mu+\lambda)\bar{u}_1 &= 0 \\ -iks(\mu+\lambda)\bar{u}_{1,2}+(2\mu+\lambda)\bar{u}_{2,22}-k^2\mu u_2 &= 0, \end{aligned} \tag{9d}$$

where $s = +1$ for $k > 0$ and $s = -1$ for $k < 0$. Solution of system (9) yields

$$\begin{aligned} \bar{u}_1 &= A\,\mathrm{e}^{-skx_2}+Bx_2\,\mathrm{e}^{-skx_2} \\ \bar{u}_2 &= -Ais\,\mathrm{e}^{-skx_2}-Bi\left(\frac{m^*}{k}+sx_2\right)\mathrm{e}^{-skx_2} \\ m^* &= (3\mu+\lambda)/(\mu+\lambda). \end{aligned} \tag{10}$$

The coefficients A and B are functions of the parameter k.

BOUNDARY CONDITIONS AND SOLUTION

The concentrated force boundary condition is prescribed with the classical Dirac delta function by Nowinski (1986). The concentrated force is $P\delta(x_1)$. The boundary conditions in their transformed limit form are

$$\bar{t}_{12} = 0, \quad \bar{t}_{22} = -P \text{ at } x_2 = 0. \tag{11}$$

We consider this condition to be incorrect for nonlocal elastic media. It brought about the stress singularity in Nowinski's solutions. We determine the nonlocal concentrated force boundary conditions according to the properties of the concentrated force in the nonlocal elastic medium [eqns (2.10), (3.15) and (2.12) derived by Wang, 1988]. The nonlocal concentrated force is now

$$F_2 = \int_{-\infty}^{\infty} \alpha(|x_1' - x_1|) P\delta(x_1')\, \mathrm{d}x_1' = \alpha(|x_1|)P \tag{12}$$

and the Fourier transforms forms of the boundary conditions are

$$\bar{t}_{12} = 0, \quad \bar{t}_{22} = -P\varepsilon(k) \text{ at } x_2 = 0, \tag{13}$$

where $\alpha(|x_1|)$ is one-dimensional nonlocal kernel and $\varepsilon(k) = \bar{\alpha}(|x_1|)$. The coefficients A and B can be derived from the combination of eqns (9b), (9c) and (13)

$$A = \frac{P}{2i(\mu+\lambda)k}, \quad B = -\frac{Ps}{2i\mu}. \tag{14}$$

The transforms of the displacements and stresses are

$$\begin{aligned} \bar{u}_1 &= -\frac{Pi}{2}\left(\frac{1}{(\mu+\lambda)k} - \frac{sx_2}{\mu}\right)\mathrm{e}^{-skx_2} \\ \bar{u}_2 &= -\frac{P}{2}\left[\frac{s}{(\mu+\lambda)k} - \frac{1}{\mu}\left(\frac{m^*s}{k} + x_2\right)\right]\mathrm{e}^{-skx_2} \end{aligned} \tag{15}$$

$$\begin{aligned} \bar{t}_{11} &= -P(1 - skx_2)\,\mathrm{e}^{-skx_2}\,\varepsilon(k) \\ \bar{t}_{22} &= -P(1 + skx_2)\,\mathrm{e}^{-skx_2}\,\varepsilon(k) \\ \bar{t}_{12} &= -Pikx_2\,\mathrm{e}^{-skx_2}\,\varepsilon(k). \end{aligned} \tag{16}$$

The transforms of the displacements do not depend on the nonlocal moduli. Their inverse transforms are also independent of the nonlocal moduli. It is easy to verify that the displacements coincide with the well-known classical expressions. This result supports the viewpoint of solving the nonlocal stress field with the classical displacement field problems (Wang, 1989b). On the contrary, it has already been mentioned that the displacements obtained by Nowinski (1986) differ from their classical counterparts.

An inverse transformation of eqn (16) yields the nonlocal stress field

$$\begin{aligned} t_{11} &= \frac{P}{\pi a^2}(I_1^+ + I_1^- - I_2^+ - I_2^-) \\ t_{22} &= \frac{P}{\pi a^2}(I_1^+ + I_1^- + I_2^+ + I_2^-) \\ t_{12} &= \frac{P}{\pi a^2}(I_3^+ + I_3^-), \end{aligned} \tag{17}$$

where

$$I_1^{\pm} = \frac{x_2}{2}\ln\frac{(a\pm x_1)^2+x_2^2}{x_1^2+x_2^2} + x_1 \arctan\frac{x_1}{x_2} - (a\pm x_1)\arctan\frac{a\pm x_1}{x_2}$$

$$I_2^{\pm} = \frac{x_2}{2}\ln\frac{x_1^2+x_2^2}{(a\pm x_1)^2+x_2^2}$$

$$I_3^{\pm} = x_2 \arctan\frac{a x_2}{(a\pm x_1)x_1 \pm x_2^2}. \tag{18}$$

This differs from the solution in classical elasticity. In contrast, the nonlocal stresses obtained by Nowinski (1986) coincide with the classical elastic solution.

We observe the following significant results.

(1) Let $a \to 0$ in eqns (17) and (18). The nonlocal stresses revert to the well-known classical solutions

$$\sigma_{11} = -\frac{P}{\pi}\frac{2x_1^2 x_2}{(x_1^2+x_2^2)^2}$$

$$\sigma_{22} = -\frac{P}{\pi}\frac{2x_2^3}{(x_1^2+x_2^2)^2}$$

$$\sigma_{12} = -\frac{P}{\pi}\frac{2x_1 x_2^2}{(x_1^2+x_2^2)^2}. \tag{19}$$

This is preconceived because in this limit the nonlocal constitutive equation (2) reverts to Hooke's law of classical elasticity.

(2) In the limit $x_2 \to 0$, we obtain

$$t_{22}(x_1, 0) = \begin{cases} \frac{P}{a}\left(\frac{|x_1|}{a} - 1\right) & |x_1| \leqslant a \\ 0 & |x_1| > a. \end{cases} \tag{20}$$

From eqn (7)

$$\bar{\alpha}(|x_1|) = \varepsilon(k) = \frac{\sin^2(ka/2)}{(ka/2)^2}; \tag{21}$$

the inverse transformation gives the one-dimensional nonlocal kernel function

$$a(|x_1|) = \begin{cases} \frac{1}{a}\left(1 - \frac{|x_1|}{a}\right) & |x_1| \leqslant a \\ 0 & |x_1| > a. \end{cases} \tag{22}$$

From eqns (12) and (22) the nonlocal concentrated force is

$$F_2 = \begin{cases} \frac{P}{a}\left(1 - \frac{|x_1|}{a}\right) & |x_1| \leqslant a \\ 0 & |x_1| > a. \end{cases} \tag{23}$$

This is a non-negative nonsingular decreasing function with a finite support that differs from the Dirac delta function so long as $a \neq 0$. Adopting different nonlocal kernels, the nonlocal concentrated force in eqn (12) can be expressed by the different function forms.

For example, another appropriate function form is the Gaussian distribution (Wang, 1988). Associating eqn (20) with eqn (23), the nonlocal stress boundary condition

$$t_{22}(x_1, 0) = -F_2(x_1) \text{ at } x_2 = 0 \tag{24}$$

is satisfied. According to Wang (1988) the nonlocal concentrated force satisfies the property

$$\int_{-\infty}^{\infty} F_2(x_1)\,\mathrm{d}x_1 = -\int_{-a}^{a} \frac{P}{a}\left(\frac{|x_1|}{a} - 1\right)\mathrm{d}x_1 = P. \tag{25}$$

The total strength of the nonlocal line force broadened is equal to that of the classical line force. From eqns (12) and (25), the representations for a concentrated load of strength P applied in the both nonlocal and classical elastic mediums are mutually correspondent.

(3) Under the limits $x_2 \to 0$, $x_1 \to 0$, we have

$$t_{22} = -P/a \text{ at } x_2 = 0, \quad x_1 = 0. \tag{26}$$

The nonlocal stress is the finite value along the line of loading. This result successfully eliminates the stress singularity which occurs in the classical stress and also the nonlocal stress derived by Nowinski (1986). Therefore, the stress singularities may be eliminated by the application of nonlocal elastic theory.

This result can be related to some physical realities. When stress makes atomic bonds break, t_{22} at the point of application of the load reaches theoretical breaking strength t_c and P reaches a theoretical cut-off strength

$$P_c = at_c. \tag{27}$$

It is interesting to find the force involved in breaking the atomic bonds. Here, we use a very rough estimation. Letting $t_c/E \sim 0.2$ (E is Young's modulus), $E \sim 50$ GNm^{-2}, $a \sim 2 \times 10^{-10}$ m, we obtain the cut-off strength $P_c \sim 2 \times 10^{-9}$ GNm^{-1}. Because the minimum line width of a technically possible line load is approximately $10 \sim 1000$ atomic spacings, the practical value of P_c may be roughly $10 \sim 1000$ times larger, or $P_c \sim 10^{-7}$ GNm^{-1} [the integrals of the stresses in eqn (17) should really be evaluated]. Therefore, the atoms of surface layers are easily cut off. This may be tested quantitatively by combining an elaborate experiment on a solid surface with rigorous theoretical calculation. It should be noted that such a result cannot be obtained through classical elastic theory since the classical maximum stresses at the edges of a load region are all singular.

DISCUSSION

The basic idea of nonlocal elasticity is that long-range (nonlocal) interactions are taken into account by a nonlocal elastic constitutive equation, therefore eliminating the stress singularities which appear in classical elasticity. The classical singular concentrated forces are smoothened and broadened into the nonsingular because of the long-range interactions or nonlocal effects characterized by nonlocal kernel functions α. The nonlocal concentrated forces should be represented by nonsingular attenuation functions and the attenuating neighbourhood is determined by adopting a nonlocal kernel in which different internal characteristic lengths (e.g. lattice parameter, granular distance, scale of a composite material structure) can be considered in the treatment of different problems. In this analysis our goal is a correctly nonlocal approach to atomic lattice effects. Equations (12) and (25) are general properties of concentrated forces in nonlocal elasticity, although a different α can be selected. It is expected that α is a δ sequence whose limit is the Dirac delta function (Eringen, 1983). Thus in the classical elasticity limit, e.g. $a \to 0$, α becomes a Dirac delta function and nonlocal elasticity reverts to classical elasticity. This guarantees that the nonlocal concentrated forces expressed by eqn (12) can revert to classical concentrated forces in the limit case. As a result of the correlation between nonlocal and classical concentrated forces, it is easy to transform concentrated force problems from classical elasticity to nonlocal elasticity. The result given in this paper can be applied as a Green function to related problems, such as finite-region load problems.

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